Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 3, pp. 373–386, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00062-X

ON THE POINCARÉ–CHETAYEV EQUATIONS†

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(*Received* 12 *January* 1994)

It is proved that, under fairly general conditions, the canonical Poincaré–Chetayev equations are Hamiltonian equations in non-canonical variables. It is shown that systems of generalized Lagrange and Hamilton equations in redundant variables, of lower order than equations that contain undetermined multipliers, as well as the Euler–Lagrange equations in quasi-coordinates, are all special cases of the Poincaré–Chetayev equations. Thus the theory of the latter extends at once to the types of systems just listed. The problem of using the Poincaré–Chetayev equations in non-holonomic dynamics is discussed.

Poincaré's remarkable idea [1] of representing the equations of motion of holonomic mechanical systems in terms of a certain transitive Lie group of infinitesimal transformations was extended by Chetayev [2–5] to the case of non-stationary constraints and dependent variables, when the transformation group is intransitive. Chetayev transformed Poincaré's equations to canonical form and developed a theory for their integration.

One important and well-known way in which the modern theory of Hamiltonian systems generalizes the classical theory is to use non-canonical coordinates [6–8], in terms of which the equations of motion often become much simpler than the clumsy and inconvenient equations in canonical coordinates q_i, p_i; this is the case, e.g. for the motion of a free rigid body. In this sense the Poincaré–Chetayev theory is extremely promising for the modern theory of Hamiltonian systems.

1. Consider a holonomic mechanical system with k degrees of freedom, whose position in space at any time t is defined by the values of the variables $x_1, \ldots, x_n$ $(n \geqslant k)$, called defining coordinates [5]. If $n=k$, the x_i are independent Lagrangian coordinates; if $n>k$ they are dependent or redundant coordinates of the system.

Suppose that certain integrable differential constraints imposed upon the system have been parametrized in some way, so that the generalized velocities may be written in the form

$$\dot{x}_i = \xi_i^s(t,x)\eta_s + \xi_i(t,x), \quad \operatorname{rank}(\xi_i^s) = k(i=1,\ldots,n;\; s=1,\ldots,k) \tag{1.1}$$

Throughout, the repeated-index summation convention will be used.

The following closed system of infinitesimal linear operators exist [4, 5]

$$X_0 = \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x_i}, \quad X_s = \xi_i^s \frac{\partial}{\partial x_i} \quad (s=1,\ldots,k,) \tag{1.2}$$

which define non-transitively acting transformations that steer the holonomic system from a position (x_i) at time t to an actual infinitesimally close position $(x_i + dx_i)$ at time $t+dt$ by the transformation

†*Prikl. Mat. Mekh.* Vol. 58, No. 3, pp. 3–16, 1994.

$$df=(X_0f+\eta_sX_sf)dt, \quad f(t,x)\in C^1 \tag{1.3}$$

and to a virtual infinitesimally close position $(x_i+\delta x_i)$ by the transformation

$$\delta f=\omega_sX_sf \tag{1.4}$$

In the case $n=k$, when $\det(\xi_i^s)\neq 0$, the transformations (1.2) act transitively.

The system of linear operators (1.2) is closed in the sense that its commutator satisfies the identity

$$[X_i,X_j]f=X_iX_jf-X_jX_if=c_{ij}^sX_sf \quad (i,j,s=0,1,\dots,k) \tag{1.5}$$

where the structure coefficients c_{ij}^s satisfy the conditions $c_{ij}^s=-c_{ji}^s$, $c_{0j}^0=0$ and may also be variable [5]: $c_{ij}^s=c_{ij}^s\,(t,x)$. If all $c_{ij}^s=\text{const}$, then system (1.2) is a Lie group of real displacements and the virtual displacement operators X_s $(s=1,\dots,k)$ form a Lie subgroup of the group of real displacements.

The parameters η_s and ω_s of the real and virtual variables, introduced by Poincaré [1], are related as follows:

$$\delta_{\eta_i}=\frac{d\omega_i}{dt}+c_{rs}^i\eta_r\omega_s+c_{0s}^i\omega_s \quad (r,s,i=1,\dots,k) \tag{1.6}$$

Poincaré, however, considered the case when all $c_{0s}^i=0$.

It is not hard to implement the parametrization (1.1) and construct a closed system of operators (1.2) if the holonomic constraints are given by some completely integrable system of Pfaffian equations [4, 5]

$$\omega_j=a_{ji}(t,x)\delta x_i=0 \quad (i=1,\dots,n;\ j=k+1,\dots,n) \tag{1.7}$$

To that end we have to choose k linear differential forms $\omega_s=a_{si}(t,\ x)\delta x_i$ $(s=1,\dots,k)$, which are independent both with respect to one another and with respect to the forms (1.7), and then to solve the resulting system of equations for dx_i

$$\delta x_i=\xi_i^s(t,x)\omega_s \quad (i,s=1,\dots,n)$$

For virtual displacements this yields expressions for the operators X_s and the relation (1.4), provided conditions (1.7) are satisfied. For real displacements constrained by completely integrable equations

$$\eta_jdt\equiv a_{ji}(t,x)dx_i+a_j(t,x)dt=0 \quad (i=1,\dots,n;\ j=k+1,\dots,n) \tag{1.8}$$

we proceed in exactly the same way, taking (1.8) together with the additional forms $\eta_sdt\equiv a_{si}(t,\ x)dx_i$ $(s=1,\dots,k)$ and dt; this yields expressions for the operators (1.2) and the relation (1.3).

As the auxiliary forms ω_i and η_idt $(i=1,\dots,k)$ may be chosen fairly arbitrarily, one can give the parameters the most convenient kinematic sense and simplify system (1.2) to a Lie group with constant structure coefficients c_{ij}^s.

Using Eqs (1.1), we can represent the kinetic energy of the system by a function $T(t,\ x_1,\dots,x_n,\ \eta_1,\dots,\eta_k)$. Assuming that active forces admitting a force function $U(t,x)$ are applied to the points of the system, as well as non-potential forces with projections F_x, F_y, F_z on the stationary axes of a Cartesian system of coordinates xyz, we introduce the function $L(t,\ x,\ \eta)=T(t,\ x,\ \eta)+U(t,\ x)$ and generalized non-potential forces

$$Q_i(t,x,\eta)=\sum(F_xX_ix+F_yX_iy+F_zX_iz) \quad (i=i,\dots,k)$$

where the summation is carried out over all points of the system.

We shall call $L(t, x, \eta)$ the generalized Lagrangian to distinguish it from the classical Lagrangian $\hat{L}(t, x, \dot{x})$, from which it is obtained by the substitution (1.1).

Chetayev [5] derived Poincaré's equations from the D'Alembert–Lagrange principle

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_i} = c_{ri}^s \eta_r \frac{\partial L}{\partial \eta_s} + c_{0i}^s \frac{\partial L}{\partial \eta_s} + X_i L + Q_i \quad (r, s, i, \ldots, k) \tag{1.9}$$

with which Eqs (1.1) must be combined in the general case. The combined system of equations of motion (1.9), (1.1) of order $k+n$ contains the same number of unknowns $x_1, \ldots, x_n, \eta_1, \ldots, \eta_k$.

Among the special cases of Eqs (1.9) are the equations formulated by Poincaré [1] for the case $n = k$, $Q_i = 0$ and

$$X_0 = \frac{\partial}{\partial t}, \quad c_{0i}^s = 0 \quad (i, s = 1, \ldots, k)$$

as well as the Lagrange equations of the second kind.

Both Poincaré and Chetayev, incidentally, assumed that the structure coefficients were constants: $c_{ij}^s = \text{const}$, in which case the system of operators (1.2) is a Lie group G. If in addition $Q_i = 0$ for all i and $X_\alpha L = 0$ $(\alpha = 0, 1, \ldots, k)$, then the Lagrangian will depend only on the parameters η_α, which may be considered as coordinates in the Lie algebra g of G; in that case Poincaré's equations (1.9) will be a closed system of differential equations in algebra g [8, 9].

Chetayev [5] pointed out, however, that Poincaré's equations are also meaningful when the coefficients c_{ij}^s are allowed to vary: $c_{ij}^s(t, x)$. We shall in fact consider this more general case.

If

$$X_0 = \partial/\partial t, \quad c_{0i}^s = 0, \quad \partial L/\partial t = 0, \quad Q_i \eta_i = 0 \quad (i, s = 1, \ldots, k) \tag{1.10}$$

then Eqs. (1.9) have a generalized energy integral

$$\eta_i \partial L/\partial \eta_i - L = \text{const}$$

Considering the case in which there are no non-potential forces Q_i, Chetayev introduced the important concept of a cyclic displacement X_i $(i = r+1, \ldots, k)$, which satisfies the conditions

$$1^0 [X_\alpha, X_i] = 0, \quad c_{\alpha i}^s = 0 \quad (\alpha, s = 0, 1, \ldots, k); \quad 2^0 X_i L = 0 \tag{1.11}$$

If cyclic displacements exist, Eqs (1.9) have first integrals

$$\partial L/\partial \eta_i = b_i = \text{const} \quad (i = r+1, \ldots, k) \tag{1.12}$$

By (1.12), the parameters $\eta_{r+1}, \ldots, \eta_k$ may be expressed as functions of $t, x, \eta_1, \ldots, \eta_r, b_{r+1}, \ldots, b_k$ and one can form a generalized Routh function

$$R(t, x_1, \ldots, x_n, \eta_1, \ldots, \eta_r, b_{r+1}, \ldots, b_k) = L - \eta_i \partial L/\partial \eta_i \tag{1.13}$$

Using the equalities

$$X_s R = X_s L, \quad X_\alpha R = 0, \quad \partial R/\partial \eta_s = \partial L/\partial \eta_s, \quad \partial R/\partial b_\alpha = -\eta_\alpha \quad (s = 1, \ldots, r; \ \alpha = r+1, \ldots, k)$$

we can write Eqs (1.9) for non-cyclic displacements as generalized Routh equations

$$\frac{d}{dt}\frac{\partial R}{\partial \eta_i} = c_{\alpha i}^{s}\eta_\alpha \frac{\partial R}{\partial \eta_s} + c_{\alpha i}^{j}\eta_\alpha b_j + c_{0i}^{s}\frac{\partial R}{\partial \eta_s} + c_{0i}^{j} b_j + X_i R \quad (i, \alpha, s = 1, \ldots, r;\ \ j = r+1, \ldots, k) \tag{1.14}$$

after integration of which the quantities η_α are determined by the relations

$$\eta_\alpha = -\partial R / \partial b_\alpha \quad (\alpha = r+1, \ldots, k) \tag{1.15}$$

Elementary integrals like (1.12) were first given by Chaplygin [10] (see also [12]).

To transform Poincaré's equation to canonical form, Chetayev replaced the variables η_i by new variables

$$y_i = \partial L / \partial \eta_i \quad (i = 1, \ldots, k) \tag{1.16}$$

and constructed a generalized Hamiltonian

$$H(t, x_1, \ldots, x_n,\ y_1, \ldots, y_k) = y_i \eta_i - L \tag{1.17}$$

It is not difficult to prove that

$$X_i H = -X_i L, \quad \eta_i = \partial H / \partial y_i \quad (i = 1, \ldots, k)$$

using which, together with (1.16), one can reduce Poincaré's equations (1.9) to the canonical Poincaré–Chetayev equations

$$\frac{dy_i}{dt} = c_{ri}^{s}\frac{\partial H}{\partial y_r} y_s + c_{0i}^{s} y_s - X_i H + Q_i, \quad \eta_i = \frac{\partial H}{\partial y_i} \quad (i, r, s = 1, \ldots, k) \tag{1.18}$$

The second group of the canonical equations (1.18) may be given another form

$$\frac{dx_j}{dt} = X_0 x_j + \frac{\partial H}{\partial y_r} X_r x_j \quad (j = 1, \ldots, n;\ \ r = 1, \ldots, k) \tag{1.19}$$

The combined system of differential equations of motion (1.18) and (1.19) is of order $k+n$ in the same unknowns $y_1, \ldots, y_k, x_1, \ldots, x_n$.

If the quantities c_{ri}^{s}, c_{0i}^{s}, Q_i and H do not depend on the coordinates x_i, the differential equations (1.18) form a closed system.

A special case of (1.18) is that of the canonical Hamilton equations when the variables x_j ($j = 1, \ldots, k = n$) are independent and the group (1.2) reduces to a permutation group; the parameters of the real displacements are the Lagrangian generalized velocities $\eta_j = \dot{x}_j$, so that the variables x_j, $y_j = \partial \hat{L} / \partial \dot{x}_j$ are the canonical coordinates.

The generalized Jacobi theorem [2–5] holds for the canonical Poincaré–Chetayev equations: if one knows a complete integral

$$V(t, x_1, \ldots, x_n,\ a_1, \ldots, a_n) + a_{n+1}, \quad \|\partial^2 V / \partial x_i \partial a_j\| \neq 0, \quad a_j = \mathrm{const} \tag{1.20}$$

of the first-order partial differential equation

$$X_0 V + H(t, x_1, \ldots, x_n,\ X_1 V, \ldots, X_k V) = 0 \tag{1.21}$$

then the solution of Eqs (1.18) and (1.19) is determined by the set of all their integrals

$$\partial V / \partial a_j = b_j = \mathrm{const}, \quad y_\alpha = X_\alpha V \quad (j = 1, \ldots, n;\ \alpha = 1, \ldots, k) \tag{1.22}$$

At first sight, the first group of integrals (1.22) yields a general solution of system (1.18), and (1.19) that depends on the $2n$ arbitrary constants a_j, b_j, while the order of the system is $k+n$. In reality, however, the solution will depend on only $k+n$ constants. In fact, since the constraints imposed on the system are represented by the completely integrable system of Pfaffian equations (1.8), the system may be reduced to the form $d\Phi_j=0$ $(j=k+1,\ldots,n)$. By (1.3), these equalities imply the relations

$$X_\alpha\Phi_j=0 \quad (\alpha=0,1,\ldots,k;\ j=k+1,\ldots,n)$$

in view of which we can add terms $c_j\Phi_j$ with arbitrary $c_j=\text{const}$ to the complete integral (1.20) of Eq. (1.21). Consequently, of the n essential constants a_i, not one of which is additive, $n-k$ will be the coefficients $c_{k+1},\ldots,c_n$, so that the complete integral (1.20) will have the following structure

$$V=W(t,x_1,\ldots,x_n,a_1,\ldots,a_k)+a_j\Phi_j+a_{n+1},\quad \frac{\partial(X_1W,\ldots,X_kW)}{\partial(a_1,\ldots,a_k)}\neq 0 \tag{1.23}$$

It follows from (1.23) that the integrals (1.22) may be written

$$\frac{\partial V}{\partial a_\alpha}=\frac{\partial W}{\partial a_\alpha}=b_\alpha,\quad \frac{\partial V}{\partial a_j}=\Phi_j=b_j,\quad y_\alpha=X_\alpha V=X_\alpha W \quad (\alpha=1,\ldots,k;\ j=k+1,\ldots,n) \tag{1.24}$$

The second group of integrals (1.24) relates to the determination of the constants b_j of the holonomic constraints, and when these are added to the first group of integrals (1.24) the solution is uniquely defined: $x_i=x_i\ (t,\ a_1,\ldots,a_k,\ b_1,\ldots,b_k,\ b_{k+1},\ldots,b_n)$, while the third group in (1.24) defines the variables y_α $(\alpha=1,\ldots,k)$.

2. We shall show that, under quite general conditions, the canonical Poincaré–Chetayev equations are Hamiltonian equations in non-canonical variables. Equations of this kind, which are frequently more convenient than Hamiltonian equations in canonical coordinates, are studied in the modern theory of Hamiltonian systems [6–8].

We shall assume throughout that $Q_i=0$ in Eqs (1.18), and $\xi_i(t,\ x)=0$ $(i=1,\ldots,n)$ in Eqs (1.1). Then $X_0=\partial/\partial t$, $c^s_{\alpha 0}=0$ $(\alpha,\ s=1,\ldots,k)$.

Define the generalized Poisson bracket of smooth functions $f(t,x,y)$ and $\varphi(t,x,y)$ by

$$(f,\varphi)=\frac{\partial\varphi}{\partial y_\alpha}X_\alpha f-\frac{\partial f}{\partial y_\alpha}X_\alpha\varphi+c^s_{\alpha i}\frac{\partial f}{\partial y_i}\frac{\partial\varphi}{\partial y_\alpha}y_s \quad (i,\alpha,s=1,\ldots,k) \tag{2.1}$$

In the special case of canonical variables x_i, $y_i=\partial\hat{L}/\partial\dot{x}_i$ $(i=1,\ldots,\ k=n)$, when system (1.2) reduces to a permutation group, formula (2.1) reduces to the classical Poisson bracket

$$(f,\varphi)=\frac{\partial f}{\partial x_i}\frac{\partial\varphi}{\partial y_i}-\frac{\partial f}{\partial y_i}\frac{\partial\varphi}{\partial x_i} \quad (i=1,\ldots,k) \tag{2.2}$$

this being the reason for the choice of sign in (2.1); Chetayev [2–4] defined generalized Poisson brackets with the opposite sign on the right of (2.1). Using (2.2), we can write the canonical Hamilton equations, as is well known, in the form

$$\dot{x}_i=(x_i,H),\quad \dot{y}_i=(y_i,H)\Leftrightarrow \dot{x}_i=\frac{\partial H}{\partial y_i},\quad \dot{y}_i=-\frac{\partial H}{\partial x_i} \quad (i=1,\ldots,k) \tag{2.3}$$

where $H(t,x_i,\ y_i)$ is the classical Hamiltonian function.

It can be seen that the generalized Poisson bracket has the same properties as the classical Poisson bracket, namely

1. they are skew-symmetric: $(f, \varphi) = -(\varphi, f)$
2. they are bilinear: $(f, \lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1(f, \varphi_1) + \lambda_2(f, \varphi_2) (\lambda_i \in R)$
3. they satisfy the Jacobi identity: $((f, \varphi), \psi) + ((\varphi, \psi), f) + ((\psi, f), \varphi) \equiv 0$
4. they obey Leibniz's rule: $(f_1 f_2, \varphi) = f_1(f_2, \varphi) + f_2(f_1, \varphi)$.

Let us assume that $f(t, x, y) = \text{const}$ is a first integral of the canonical equations (1.18) and (1.19). Then, using definition (2.1), we have the identity

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial y_\alpha} X_\alpha f + \frac{\partial f}{\partial y_\alpha}\left(c_{i\alpha}^s \frac{\partial H}{\partial y_i} y_s - X_\alpha H \right) = \frac{\partial f}{\partial t} + (f, H) \equiv 0 \tag{2.4}$$

The following generalization of Poisson's theorem [2–4] is true: If $\varphi(t, x, y) = a$ and $\psi(t, x, y) = b$ are the first two integrals of Eqs (1.18) and (1.19), then $(\varphi, \psi) = c$ will be the third first integral of those equations.

We will now prove that the canonical Poincaré–Chetayev equations (1.18) and (1.19) may be expressed in the form

$$\frac{dy_i}{dt} = (y_i, H), \quad \frac{dx_j}{dt} = (x_j, H) \quad (i = 1, \dots, k;\ j = 1, \dots, n) \tag{2.5}$$

where $H(t, x, y)$ is the generalized Hamiltonian function (1.17).

Indeed, by definition (2.1)

$$(y_i, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha y_i - \frac{\partial y_i}{\partial y_\alpha} X_\alpha H + c_{\alpha j}^s \frac{\partial y_i}{\partial y_j} \frac{\partial H}{\partial y_\alpha} y_s = -X_i H + c_{\alpha i}^s \frac{\partial H}{\partial y_\alpha} y_s$$

$$(x_j, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha x_j - \frac{\partial x_j}{\partial y_\alpha} X_\alpha H + c_{\alpha i}^s \frac{\partial x_j}{\partial y_i} \frac{\partial H}{\partial y_\alpha} y_s = \frac{\partial H}{\partial y_\alpha} X_\alpha x_j \tag{2.6}$$

since $X_\alpha y_i = 0$, $\partial y_i / \partial y_\alpha = \delta_{i\alpha}$, $\partial x_j / \partial y_\alpha = 0$ by virtue of the fact that the variables x_j are independent of y_i ($i, \alpha = 1, \dots, k$; $j = 1, \dots, n$) and vice versa, and that the variables y_i are also independent; $\delta_{i\alpha}$ is the Kronecker delta.

Comparing the right-hand sides of Eqs (1.18) and (1.19) with formulae (2.6), we confirm the correctness of (2.5). This implies that the canonical Poincaré–Chetayev equations are Hamiltonian equations in non-canonical variables for which, consequently, the results of [2–5] are applicable.

Example 2.1. We will develop the Poincaré–Chetayev equations of motion for a heavy rigid body with one fixed point, for which the kinetic and potential energies are respectively $T = 1/2 A_i \omega_i^2$ and $V = Mgx_i^0 \gamma_i$, where ω_i are the projections of the angular velocity, γ_i are the cosines of the angles between the vertical and the principal axes of inertia, and x_i^0 are the coordinates of the centre of mass.

As the defining coordinates x_i and parameters of the real displacements η_i we take γ_i and ω_i, respectively ($i = 1, 2, 3$), where γ_i satisfy the Poisson's equations

$$d\gamma_1 / dt = \omega_3 \gamma_2 - \omega_2 \gamma_3 \quad (1\ 2\ 3)$$

Let $f(\gamma_1, \gamma_2, \gamma_3)$ be a function whose derivative is

$$\frac{df}{dt} = \omega_1 \left(\frac{\partial f}{\partial \gamma_2} \gamma_3 - \frac{\partial f}{\partial \gamma_3} \gamma_2 \right) + \omega_2 \left(\frac{\partial f}{\partial \gamma_3} \gamma_1 - \frac{\partial f}{\partial \gamma_1} \gamma_3 \right) + \omega_3 \left(\frac{\partial f}{\partial \gamma_1} \gamma_2 - \frac{\partial f}{\partial \gamma_2} \gamma_1 \right)$$

Hence we obtain expressions for the operators (1.2)

$$X_1 f = \gamma_3 \frac{\partial f}{\partial \gamma_2} - \gamma_2 \frac{\partial f}{\partial \gamma_3} \quad (1\ 2\ 3) \tag{2.7}$$

whose commutator is

$$[X_1, X_2] = X_3 f \quad (1\ 2\ 3) \tag{2.8}$$

Consequently, the non-vanishing structure constants will be $c_{12}^2 = c_{23}^1 = c_{31}^2 = 1$, $c_{21}^3 = c_{32}^1 = c_{13}^2 = -1$. Poincaré's equations (1.9) reduce to Euler's equations

$$A_1 d\omega_1 / dt = (A_2 - A_3)\omega_2\omega_3 + Mg(x_3^0\gamma_2 - x_2^0\gamma_3) \quad (1\ 2\ 3) \tag{2.9}$$

to which we must add Poisson's equations.

We replace ω_i by the variables $y_i = \partial T / \partial \omega_i$ and consider the function $H = y_i^2/(2A_i) + Mgx_i^0\gamma_i$. The Poincaré–Chetayev equations (1.18) and (1.19) take the form of Hamiltonian equations

$$\frac{dy_1}{dt} = \frac{A_2 - A_3}{A_2 A_3} y_2 y_3 + Mg(x_3^0\gamma_2 - x_2^0\gamma_3), \quad \frac{d\gamma_1}{dt} = \gamma_2 \frac{y_3}{A_3} - \gamma_3 \frac{y_2}{A_2} \quad (1\ 2\ 3) \tag{2.10}$$

if we note that the right-hand sides of these equations are Poisson brackets

$$(y_1, H) = -X_1 H + c_{\alpha 1}^s \frac{\partial H}{\partial y_\alpha} y_s, \quad (\gamma_1, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha \gamma_1 \quad (1\ 2\ 3) \tag{2.11}$$

We note, among other things, that the representation (2.1) of the Poisson bracket for the functions F and H of the variables y_i and γ_i is more compact compared than its representation as a sum of vector-scalar products [12] of three vectors

$$-y \cdot (\nabla_y F \times \nabla_y H) - \gamma \cdot (\nabla_y F \times \nabla_\gamma H + \nabla_\gamma F \times \nabla_y H)$$

where the symbols ∇_y and ∇_γ denote the gradients with respect to y and γ, and F is successively equated to the projections y_i and γ_i respectively.

3. We will now derive the generalized Lagrange and Hamilton equations in term of dependent coordinates.

Suppose that the constraints are represented by differential equations

$$\dot{x}_j = b_{j\alpha}(t, x)\dot{x}_\alpha + b_j(t, x) \tag{3.1}$$

which constitute a completely integrable Pfaffian system. Here and throughout this section, α, $i = 1, \ldots, k$; $j = k+1, \ldots, n$.

The virtual displacements are defined by the equations

$$\delta x_j = b_{j\alpha}\delta x_\alpha \tag{3.2}$$

As parameters of the real and virtual displacements we take [5] $\eta_\alpha = \dot{x}_\alpha$ and $\omega_\alpha = \delta x_\alpha$ respectively. Using (3.1), we find expressions for the operators of the group (1.2)

$$X_0 f = \frac{\partial f}{\partial t} + b_j \frac{\partial f}{\partial x_j}, \quad X_\alpha f = \frac{\partial f}{\partial x_\alpha} + b_{j\alpha} \frac{\partial f}{\partial x_j} \tag{3.3}$$

whose commutator vanishes because Eqs (3.1) and (3.2) are integrable. Consequently, all the structure constants of the group (3.3) vanish: $c_{\kappa\beta}^s = 0$ (κ, β, $s = 0, 1, \ldots, k$), i.e. the operators (3.3) form an Abelian group. Poincaré's equations (1.9) in that case become

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_\alpha} - X_\alpha L = Q_\alpha$$

or, in view of (3.3)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\alpha} - \frac{\partial L}{\partial x_\alpha} - b_{j\alpha}\frac{\partial L}{\partial x_j} = Q_\alpha \quad (L = L(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_k)) \tag{3.4}$$

Equations (3.4) and (3.1), taken together, form a system of differential equations of order $n+k$ in the same number of unknowns $x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_k$.

Equations (3.4) are generalized Lagrange equations in the dependent (redundant) coordinates; these equations do not involve the reactions to the constraints. They are more convenient than the Lagrange equations with undetermined multipliers λ_j

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{x}_i} - \frac{\partial \hat{L}}{\partial x_i} = Q_i - b_{ji}\lambda_j, \quad \frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{x}_j} - \frac{\partial \hat{L}}{\partial x_j} = Q_j + \lambda_j \quad \hat{L} = \hat{L}(t, x_1, \ldots, x_n, \ \dot{x}_1, \ldots, \dot{x}_n) \tag{3.5}$$

whose order is $2n > n+k$, considered together with equations (3.1), which are of order k. It is true that Eqs (3.5) and (3.1) enable one to determine not only $x_1, \ldots, x_n$ but also λ_j ($j = k+1, \ldots, n$), and together with them the reactions of the constraints (3.1); but for large n they are not very tractable.

It can be shown, however, that the elimination of λ_j from Eqs (3.5) and the use of (3.1) lead to Eqs (3.4); we shall not go into details.

If we replace the parameters $\eta_\alpha = \dot{x}_\alpha$ by the variables $y_\alpha = \partial L / \partial \dot{x}_\alpha$ and form the function $H(t, x_1, \ldots, x_n, y_1, \ldots, y_k) = y_\alpha \dot{x}_\alpha - L$ (summation over the repeated index α from 1 to k), then Eqs (3.4) transform to the canonical Poincaré–Chetayev equations (1.18)

$$\frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha} - b_{j\alpha}\frac{\partial H}{\partial x_j} + Q_\alpha, \quad \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha} \tag{3.6}$$

which are generalized Hamilton equations in the dependent coordinates. Together with these equations we must consider Eqs (3.1), rewritten as

$$\frac{dx_j}{dt} = b_{j\alpha}\frac{\partial H}{\partial y_\alpha} + b_j \tag{3.7}$$

thus obtaining a combined system of differential equations of order $n+k$ in the same number of unknowns $x_1, \ldots, x_n, y_1, \ldots, y_k$.

Thus, the generalized Lagrange and Hamilton equations in redundant coordinates are special cases of the Poincaré–Chetayev equations.

It is interesting to compare Suslov's theorem [13, chap. XLIII] with the generalized Jacobi theorem. Suslov considered the Hamilton equations with multipliers conjugate to Eqs (3.5), and instead of (1.21) he obtained the following partial differential equation [13, formula (43.25)]

$$\frac{\partial V}{\partial t} - \Lambda_j \frac{\partial f_j}{\partial t} + H\left(t, x_1, \ldots, x_n, \ \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0 \tag{3.8}$$

where $f_j(t, x) = c_j$ are the integrated equations of the constraints (3.1), $\Lambda_j = -\int \lambda_j dt$ are impulse factors; he proved that if one knows a compete integral of equation (3.8), then the equations

$$\frac{\partial V}{\partial a_s} = b_s, \quad \frac{\partial V}{\partial x_s} = p_s + \Lambda_j \frac{\partial f_j}{\partial x_s} \quad (s = 1, \ldots, n)$$

are integrals of the Hamilton equations conjugate to Eqs (3.5). Proposing to eliminate the impulse factors with the help of the differentiated equations of the constraints, Suslov then obtained an equation of the form (1.21), taking into account (3.3), and integrals (1.22), just as the elimination of λ_j from Eqs (3.5) led to Eqs (3.4).

Example 3.1. Let us consider Suslov's Example 134 in his notation [13]: two heavy particles of masses m_1 and m_2 with coordinates y_1, z_1 and y_2, z_2 respectively, are moving in the yz plane, subject to the constraints

$$m_1\dot{y}_1 + m_2\dot{y}_2 = 0, \quad (y_1 - y_2)(\dot{y}_1 - \dot{y}_2) + (z_1 - z_2)(\dot{z}_1 - \dot{z}_2) = 0 \qquad (M = m_1 + m_2)$$

The position of the system is determined by the coordinates y_c, z_c of the centre of mass and by the quantities $\eta = y_1 - y_2$ and $\varphi = \operatorname{arctg}[(z_1 - z_2)/(y_1 - y_2)]$, in terms of which the integrated equations of the constraints become

$$y_c = c_1 = \text{const}, \quad \eta \sec\varphi = c_2 = \text{const}$$

If we take the quantities $\dot{z}_c$ and $\dot{\varphi}$ as the parameters of the real displacements, the operators (1.2) will be

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial z_c}, \quad X_2 = \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \eta}(\eta \operatorname{tg}\varphi)$$

and Eq. (1.21) will be

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{1}{M}\left(\frac{\partial V}{\partial z_c}\right)^2 + \frac{M}{m_1 m_2} \frac{\cos^2\varphi}{\eta^2}\left(\frac{\partial V}{\partial \varphi} - \eta \operatorname{tg}\varphi \frac{\partial V}{\partial \eta}\right)^2\right] + Mgz_c = 0$$

which is identical with an equation of the type (3.8) after the impulse factors of the constraints are eliminated [13, p. 471].

4. Let us consider the equations of motion of holonomic systems in quasi-coordinates, which have been attracting attention in the literature for many years (e.g. [14–17]). We will show that these equations are a special case of the Poincaré–Chetayev equations.

Let x_i and $\dot{x}_i$ $(i = 1, \ldots, k)$ be the independent Lagrangian coordinates and velocities of a holonomic system, $t = x_0$ the time, and $\dot{x}_0 = 1$. As parameters of the real displacements we take the quasi-velocities η_s, which are related to x_i through the non-integrable equations

$$\eta_s = a_{si}(x)\dot{x}_i \quad (s, i = 0, 1, \ldots, k), \quad \det(a_{si}) \neq 0 \tag{4.1}$$

Then [16]

$$\dot{x}_i = b_{is}(x)\eta_s \tag{4.2}$$

where $\eta_0 = 1$, $a_{0i} = b_{0i} = \delta_{0i}$, $b_{i0} = -b_{is}a_{s0}$, $a_{si}b_{ir} = a_{is}b_{ri} = \delta_{sr}$ $(i, r, s = 0, 1, \ldots, k)$.

We now introduce the differentials of the quasi-coordinates

$$d\pi_s = \eta_s dt = a_{si} dx_i \quad (i, s = 0, 1, \ldots, k) \tag{4.3}$$

in terms of which the differentials of the coordinates x_i may be written in the form

$$dx_i = b_{is} d\pi_s \quad (d\pi_0 = dt) \tag{4.4}$$

We introduce *ad hoc* notation for the derivatives of a function $f(x) \in C^1$ with respect to the quasi-coordinates

$$\frac{\partial f}{\partial \pi_s} = b_{rs} \frac{\partial f}{\partial x_r}, \quad \frac{\partial f}{\partial x_r} = a_{sr} \frac{\partial f}{\partial \pi_s} \quad (r, s = 0, 1, \ldots, k) \tag{4.5}$$

using which, together with the parametrization (4.2), we construct operators

$$X_s f = \frac{\partial f}{\partial \pi_s} = b_{rs} \frac{\partial f}{\partial x_r} \quad (r, s = 0, 1, \ldots, k) \tag{4.6}$$

with commutator

$$[X_i, X_j] f = \frac{\partial^2 f}{\partial \pi_i \partial \pi_j} - \frac{\partial^2 f}{\partial \pi_j \partial \pi_i} = c_{ij}^s \frac{\partial f}{\partial \pi_s} \quad (i, j, s = 0, 1, \ldots, k) \tag{4.7}$$

where the structure coefficients are

$$c_{ij}^s = a_{sr} \left(\frac{\partial b_{rj}}{\partial x_\alpha} b_{\alpha i} - \frac{\partial b_{ri}}{\partial x_\alpha} b_{\alpha j} \right) = \left(\frac{\partial a_{sr}}{\partial x_\alpha} - \frac{\partial a_{s\alpha}}{\partial x_r} \right) b_{ri} b_{\alpha j} \tag{4.8}$$

$$(\alpha, i, j, r, s = 0, 1, \ldots, k)$$

It is obvious that the infinitesimal operators (4.6) form a closed system. Consequently, Poincaré's equations retain the form (1.9) even in terms of quasi-coordinates, if one takes (4.6) and (4.8) into account

$$\frac{d}{dt} \frac{\partial L}{\partial \eta_i} = c_{\alpha i}^s \eta_\alpha \frac{\partial L}{\partial \eta_s} + c_{0i}^s \frac{\partial L}{\partial \eta_s} + \frac{\partial L}{\partial \pi_i} + Q_i \quad (\alpha, i, s = 1, \ldots, k) \tag{4.9}$$

Combining Eqs (4.9) with the kinematic relations (4.2), we obtain a system of $2k$ first-order ordinary differential equations, each in the same number of unknowns $\eta_1, \ldots, \eta_k, x_1, \ldots, x_k$.

Introducing the three-index Boltzmann symbols

$$\gamma_{ij}^s = c_{ji}^s, \quad \gamma_{0j}^s = \varepsilon_j^s = c_{j0}^s \quad (i, j, s = 1, \ldots, k)$$

we conclude that, by (4.6), the Euler–Lagrange equations (8.1.5) of [16] in quasi-coordinates are identical with Eqs (4.9) when $P_i = Q_i$. Thus, the Euler–Lagrange equations are a special case of Poincaré's equations, when the quasi-velocities (4.1) are taken as the parameters η_s of the real displacements.

On changing from the variables η_i to the variables $y_i = \partial L / \partial \eta_i$ $(i = 1, \ldots, k)$ the equations of motion (4.9) take the form of the Poincaré–Chetayev equations (1.18), i.e.

$$\frac{dy_i}{dt} = c_{\alpha i}^s \frac{\partial H}{\delta y_\alpha} y_s + c_{0i}^s y_s - \frac{\partial H}{\partial \pi_i} + Q_i, \quad \eta_i = \frac{\partial H}{\partial y_i} \quad (\alpha, i, s = 1, \ldots, k) \tag{4.10}$$

which are the canonical form of the Euler–Lagrange equations. Together with Eqs (4.10) we must consider Eqs (4.2), rewritten in the form

$$\dot{x}_i = b_{is}\partial H / \partial y_s \quad (i, s = 0, 1, \ldots, k) \tag{4.11}$$

Thus, we obtain a system of $2k$ differential equations in $2k$ unknowns $y_1, \ldots, y_k, x_1, \ldots, x_k$ (Eq. (4.11) for $i = 0$ reduces to the identity $1 = 1$, since $\dot{x} = b_{00} = \eta_0 = \partial H / \partial y_0 = 1$).

Clearly, the theory of the Poincaré–Chetayev equations is also applicable to the Euler–Lagrange equations in quasi-coordinates.

Example 4.1. For a heavy rigid body with one fixed point (Example 2.1), the Lagrangian coordinates x_i and quasi-velocities η_i will be the Euler angles $x_1 = \theta$, $x_2 = \psi$, $x_3 = \varphi$ and the projections ω_i $(i = 1, 2, 3)$ of the angular velocity on the principal axes of inertia, so that the kinetic and potential energies become

$$T = \tfrac{1}{2} A_i \omega_i^2, \quad V = Mg(x_1^0 \sin\theta \sin\varphi + x_2^0 \sin\theta \cos\varphi + x_3 \cos\theta)$$

Using relations of the type (4.2)

$$\dot{\theta} = \omega_1 \cos\varphi - \omega_2 \sin\varphi, \quad \dot{\psi} = (\omega_1 \sin\varphi + \omega_2 \cos\varphi)/(\sin\theta), \quad \dot{\varphi} = \omega_3 - \mathrm{ctg}\,\theta(\omega_1 \sin\varphi + \omega_2 \cos\varphi) \tag{4.12}$$

we construct the operators of a transitive Lie group

$$\begin{aligned} X_1 f &= \frac{\partial f}{\partial \pi_1} = b_{r1}\frac{\partial f}{\partial x_r} = \cos\varphi \frac{\partial f}{\partial \theta} + \frac{\sin\varphi}{\sin\theta}\frac{\partial f}{\partial \psi} - \mathrm{ctg}\,\theta \sin\varphi \frac{\partial f}{\partial \varphi} \\ X_2 f &= \frac{\partial f}{\partial \pi_2} = b_{r2}\frac{\partial f}{\partial x_r} = -\sin\varphi \frac{\partial f}{\partial \theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial f}{\partial \psi} - \mathrm{ctg}\,\theta \cos\frac{\partial f}{\partial \varphi} \\ X_3 f &= \frac{\partial f}{\partial \pi_3} = b_{r3}\frac{\partial f}{\partial x_r} = \frac{\partial f}{\partial \varphi} \end{aligned} \tag{4.13}$$

with commutator (2.8). The non-vanishing structure constants are $c_{12}^3 = c_{23}^1 = c_{31}^2 = 1$, $c_{21}^3 = c_{32}^1 = c_{13}^2 = -1$, as in Example 2.1.

The Poincaré equations have the form of (2.9), on the right-hand side of which γ_i must be replaced by

$$\gamma_i = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta \cos\varphi, \quad \gamma_3 = \cos\theta \tag{4.14}$$

Equations (2.9) are completed by adding Eqs (4.12).

The canonical Poincaré–Chetayev equations have the form of the first group of equations (2.10), taking into account (4.13) and Eqs (4.12) with ω_i in the latter replaced by y_i / A_i $(i = 1, 2, 3)$ where these equations take the form of Hamilton equations

$$\dot{\theta} = (\theta, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha \theta, \quad \dot{\psi} = (\psi, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha \psi, \quad \dot{\varphi} = (\varphi, H) = \frac{\partial H}{\partial y_\alpha} X_\alpha \varphi, \quad (\alpha = 1, 2, 3)$$

$$H = y_\alpha^2 / (2A_\alpha) + Mg(x_1^0 \sin\theta \sin\varphi + x_2^0 \sin\theta \cos\varphi + x_3^0 \cos\theta)$$

5. In conclusion, we will briefly consider the application of the Poincaré–Chetayev equations to non-holonomic dynamics. This question was previously considered in [9, 18, 19], but the results of Section 4 provide a new approach.

The Euler–Lagrange equations in quasi-coordinates combine the equations of motion for both holonomic and non-holonomic equations [14–17]. Consequently, the same is true of the Poincaré–Chetayev equations. Indeed, retaining the notation of Section 4, let us assume that the system under consideration is subject to non-integrable constraints of the form

$$\eta_s \equiv a_{sr}\dot{x}_r = 0, \quad a_{sr} = a_{sr}(x_j), \quad \mathrm{rank}(a_{sr}) = k - m \quad (r, j = 0, 1, \ldots, k; \quad s = m+1, \ldots, k) \tag{5.1}$$

To Eqs (5.1) we add arbitrary linear forms

$$\eta_i = a_{ir}x_r, \quad a_{ir} = a_{ir}(x_j) \quad (i = 1, \ldots, m) \tag{5.2}$$

but such that $\det(a_{sr}) \neq 0$ $(r, s = 0, 1, \ldots, k)$ in particular, the quantities η_i $(i = 1, \ldots, m)$ may be generalized velocities $\dot{x}_i$. Adding $\eta_0 = 1$ to the relations (5.1) and (5.2) and solving for $\dot{x}_r$ $(r = 0, 1, \ldots, k)$ we obtain (4.2).

Virtual variations of the quasi-coordinates $\pi_r = \eta_r dt$ are determined by equalities $\pi_r = \omega_r = a_{rj}\delta x_j$ $(r = 1, \ldots, k)$, where, by Eqs (5.1), we have constraints $\delta\pi_s = 0$ $(s = m+1, \ldots, k)$. Using the D'Alembert–Lagrange principle, we conclude that, unlike the results of Section 4, if we are considering a non-holonomic system of m Poincaré equations of the form (4.9)

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_i} = c^r_{\alpha i}\eta_\alpha \frac{\partial L}{\partial \eta_r} + c^r_{0i}\frac{\partial L}{\partial \eta_r} + \frac{\partial L}{\partial \pi_i} + Q_i \quad (i, \alpha = 1, \ldots, m; \quad r = 1, \ldots, k) \tag{5.3}$$

The structure coefficients c^r_{ij} are also determined by (4.8), but with the indices i, j varying from 0 to m.

Equations (5.3), together with the constraint equations (5.1) and the relations (5.2), form a system of $k+m$ equations of motion of a non-holonomic system in quasi-coordinates with the same number of unknowns $x_1, \ldots, x_k, \eta_1, \ldots, \eta_m$. It should be stressed that the generalized Lagrangian $L(t, x_1, \ldots, x_k, \eta_1, \ldots, \eta_k)$ appearing in (5.3) may depend on all k quasi-velocities η_r, and it is necessary to use the constraint equations (5.1), $\eta_s = 0$ $(s = m+1, \ldots, k)$, only after setting up Eqs (5.3) [16, 17].

Note that, by the method described in [5] to determine the reactions to constraints, the remaining $k-m$ equations (4.9), with the terms $b_{ij}R_i$ added to their right-hand sides, enable us to find the reactions R_i to the constraints (5.1). If we free the system from the constraints (5.1), replacing their effects by the reactions R_i $(i = 1, \ldots, k)$, the result will be a holonomic system, to which equations of type (4.9) are applicable. Since the constraints are assumed to be ideal, the work done by their reactions in virtual displacements will vanish

$$R_i\delta x_i = R_i b_{ij}\delta\pi_j = 0 \quad (i = 1, \ldots, k; \quad j = 1, \ldots, m)$$

Hence, since $\delta\pi_j$ is arbitrary, it follows [9] that $R_i b_{ij} = 0$ $(j = 1, \ldots, m)$ which implies that the first m equations of motion of the "freed" system are Eqs (5.3), and the remaining $k-m$ equations

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_s} = c^r_{\alpha s}\eta_\alpha \frac{\partial L}{\partial \eta_r} + c^r_{0s}\frac{\partial L}{\partial \eta_r} + \frac{\partial L}{\partial \pi_s} + Q_s + b_{is}R_i, \quad (s = m+1, \ldots, k) \tag{5.4}$$

in view of (5.1), enable us to determine the reactions R_i, provided that $\mathrm{rank}(b_{is}) \neq 0$.

Note that if $Q_i = 0$ Eqs (5.3) are equivalent to Eqs (3.14) of [18] and (1.13) of [19], but they are slightly simpler thanks to the choice of the quasi-velocities η_s, which vanish because of the equations of the non-holonomic constraints (5.1). Let us replace the kinetic energy $T(t, x, \eta_1, \ldots, \eta_k)$ of the holonomic system, occurring in the function $L(t, x, \eta)$ in Eqs (5.3), by the kinetic energy $\theta(t, x, \eta_1, \ldots, \eta_m)$ of the non-holonomic system with constraints (5.1). Obviously

$$\theta(t, x, \eta_1, \ldots, \eta_m) = T(t, x, \eta_1, \ldots, \eta_m, 0, \ldots, 0).$$

Consequently, if $\eta_s = 0$ $(s = m+1, \ldots, k)$, then

$$\frac{\partial L}{\partial \eta_i}=\frac{\partial \theta}{\partial \eta_i},\quad \frac{\partial L}{\partial \pi_i}=\frac{\partial(\theta+U)}{\partial \pi_i},\quad \frac{\partial L}{\partial \eta_r}=\left(\frac{\partial T}{\partial \eta_r}\right)_{\eta_s=0},\quad (i=1,\ldots,m;\; r=m+1,\ldots,k)$$

and Eqs (5.3) become

$$\frac{d}{dt}\frac{\partial \theta}{\partial \eta_i}=(c_{\alpha i}^s\eta_\alpha+c_{0i}^s)\frac{\partial \theta}{\partial \eta_s}+(c_{\alpha i}^r\eta_\alpha+c_{0i}^r)\left(\frac{\partial T}{\partial \eta_r}\right)_{\eta_s=0}+\frac{\partial(\theta+U)}{\partial \pi_i}+Q_i \tag{5.5}$$

$$(\alpha,i,s=1,\ldots,m;\; r=m+1,\ldots,k)$$

as in the case of Eqs (1.13) of [19]. Here $(\partial T/\partial \eta_r)_{\eta_s=0}$ denote the expressions $\partial T/\partial \eta_r$ when $\eta_s=0$ $(r,\ s=m+1,\ldots,k)$.

In the special case in which the parameters η_i of (5.2) are the generalized velocities $\dot{x}_i=\eta_i$ $(i=1,\ldots,m)$, i.e. when $a_{ir}=\delta_{ir}(i,\ r=1,\ldots,m)$, all the structure coefficients vanish: $c_{\alpha i}^r=0$ for $r\leqslant m$ [14], and equations (5.3) become

$$\frac{d}{dt}\frac{dL}{\partial \dot{x}_i}=c_{\alpha i}^r\dot{x}_\alpha\frac{\partial L}{\partial \eta_r}+c_{0i}^r\frac{\partial L}{\partial \eta_r}+\frac{\partial L}{\partial \pi_i}+Q_i \tag{5.6}$$

$$(i,\alpha=1,\ldots,m;\; r=m+1,\ldots,k;\; L=L(t,x_1,\ldots,x_k,\ \dot{x}_1,\ldots,\dot{x}_m,\ \eta_{m+1},\ldots,\eta_k))$$

If we replace the parameters η_i by variables $y_i=\partial L/\partial \eta_i$ $(i=1,\ \ldots,\ k)$, the equations of motion (5.3) of the non-holonomic system take the form of the canonical Poincaré–Chetayev equations

$$\frac{dy_i}{dt}=c_{\alpha i}^s\frac{\partial H}{\partial y_\alpha}y_s+c_{0i}^s y_s-\frac{\partial H}{\partial \pi_i}+Q_i \tag{5.7}$$

$$\eta_i=\partial H/\partial y_i\quad (i,\alpha=1,\ldots,m;\quad s=1,\ldots,k)$$

to which we must add the constraint equations (5.1) and relations (4.2), rewritten in the form

$$\frac{\partial H}{\partial y_s}=0\quad (s=m+1,\ldots,k),\quad \dot{x}_i=b_{ij}\frac{\partial H}{\partial y_j},\quad (i=1,\ldots,k,\ j=0,1,\ldots,m) \tag{5.8}$$

Equations (5.7) and (5.8) form a system of $2k+m$ equations in the same number of unknowns $x_1,\ldots,x_k,\ y_1,\ldots,y_k,\ \eta_1,\ldots,\eta_m$ [17].

Example 5.1. Working from Eqs (5.6), let us derive the equations of motion in Voronets' form [20] for a system with Lagrangian coordinates $x_1,\ldots,x_n$ and non-integrable constraints

$$\dot{x}_s=\alpha_{si}(t,x)\dot{x}_i+\alpha_s(t,x),\quad (i=1,\ldots,m;\; s=m+1,\ldots,n) \tag{5.9}$$

Set $\eta_i=\dot{x}_i$, $x_0=t$, $\dot{x}_0=1$, $\eta_s=\dot{x}_s-\alpha_{si}\dot{x}_i$, $\alpha_{s0}=\alpha_s$ $(i=0,\ 1,\ \ldots,\ m;\ s=m+1,\ \ldots,\ n)$, so that $\dot{x}_i=\eta_i$, $\dot{x}_s=\eta_s+\alpha_{si}\eta_i$ and, in view of (4.1), (4.2), we have the relations

$$a_{ij}=b_{ij}=\delta_{ij},\quad a_{is}=b_{is}=0,\quad b_{si}=-a_{si}=\alpha_{si},\quad a_{sr}=b_{sr}=\delta_{sr}$$

$$(i,j=0,1,\ldots,m;\quad s,r=m+1,\ldots,n)$$

according to which, by formulae (4.8)

$$c^r_{ji} = \frac{\partial \alpha_{ri}}{\partial x_j} - \frac{\partial \alpha_{rj}}{\partial x_i} + \frac{\partial \alpha_{ri}}{\partial x_k}\alpha_{kj} - \frac{\partial \alpha_{rj}}{\partial x_k}\alpha_{ki}.$$

$$c^r_{0i} = \frac{\partial \alpha_{ri}}{\partial t} - \frac{\partial \alpha_r}{\partial x_i} + \frac{\partial \alpha_{ri}}{\partial x_k}\alpha_k - \frac{\partial \alpha_r}{\partial x_k}\alpha_{ki} \qquad (5.10)$$

$$(i, j = 1, \ldots, m;\ k, r = m+1, \ldots, n)$$

Noting (5.9) and (5.10), we conclude that Eqs (5.6) now take the form of Voronets' equations [20]

$$\frac{d}{dt}\frac{\partial \theta}{\partial \dot{x}_i} = \frac{\partial(\theta + U)}{\partial x_i} + \alpha_{ri}\frac{\partial(\theta+U)}{\partial x_r} + (c^r_{ji}\dot{x}_j + c^r_{0i})\left(\frac{\partial L}{\partial \dot{x}_r}\right) + Q_i \qquad (5.11)$$

$$(i, j = 1, \ldots, m;\ r = m+1, \ldots, n)$$

Equations (5.11) and (5.9) combined form a system of n differential equations, whose general solution depends on $n+m$ arbitrary constants.

In the special case in which the function $L(t, x, \dot{x}_m, \eta_{m+1}, \ldots, \eta_n)$, the forces Q_i and also the coefficients of the constraints (5.9) do not depend explicitly on the coordinates x_r $(r = m+1, \ldots, n)$, Eqs (5.11) are identical with the closed system of m Chaplygin equations [17] in the unknowns $x_1, \ldots, x_m$.

I wish to thank L. M. Markhashov for useful comments.

The research reported here was financially supported by the Russian Fund for Fundamental Research (93-013-16242).

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Translated by D.L.